Mathematics 1c, Spring 2008
Solutions to the Midterm Exam

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Print Your Name:
Your Section:

- This exam has five questions.
- You may take ***three hours***; there is no credit for overtime work
- ***No aids*** (including notes, books, calculators etc.) are permitted.
- The exam must be turned in by ***noon on Wednesday, May 7.*** Please turn in your exam to the Math 1C Exam Drop Slot outside Room 255 Sloan.
- All 5 questions should be answered on this exam, using the backs of the sheets or appended pages if necessary.
- Show all your work and justify all claims using plain English.
- Each question is worth 20 points.
- The exam has pages numbered 1–6, including this cover sheet.
- **Good Luck !!**

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1.

(a) Compute the orthogonal projection of the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ onto the subspace $W \subset \mathbb{R}^3$ that is spanned by the two vectors $\mathbf{i}$ and $\mathbf{i}+\mathbf{j}-\mathbf{k}$.

(b) Let V be the vector space of polynomials of degree 3. Define the operator $T : V \to V$ by $T(f) = xf'$. Show T is diagonalizable.

(c) Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix. Assemble a 3×3 matrix B using the following notation:

$$B = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that

$$B^2 = \begin{bmatrix} A^2 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Compute

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{100}$$

and

$$\begin{bmatrix} 2 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{101}$$

Solution.

(a) First of all, orthonormalize the two vectors $\mathbf{i}$ and $\mathbf{i}+\mathbf{j}-\mathbf{k}$ to produce the orthonormal basis of W given by $\mathbf{v}_1 = \mathbf{i}$ and $\mathbf{v}_2 = (\mathbf{j}-\mathbf{k})/\sqrt{2}$. Letting $\mathbf{u} = \mathbf{i}+\mathbf{j}+\mathbf{k}$, the orthogonal projection is then given by

$$(\mathbf{u}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{u}\cdot\mathbf{v}_2)\mathbf{v}_2 = \mathbf{i} + 0(\mathbf{j}-\mathbf{k})/\sqrt{2} = \mathbf{i}$$

(b) The space V has dimension 4 and by inspection, there are four distinct eigenvalues $0, 1, 2, 3$ with associated eigenvectors given by the polynomials $1, x, x^2, x^3$. Since their are 4 distinct eigenvalues (or directly the eigenvectors form a basis), the operator is diagonalizable.

(c) One verifies this by a simple direct calculation.

(d) If we let $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ and construct B as in part (c), we get the given matrix. As in (c), raised to the 100th power, it is

$$B^{100} = \begin{bmatrix} A^{100} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and similarly

$$B^{101} = B$$

The computation of A^{100} and A^{101} was done explicitly in lecture. This can be done in a couple of ways.

Method 1. Perhaps the simplest way is to directly compute that $A^2 =$ Identity and thus, $A^{100} =$ Identity and $A^{101} = A$.

Method 2–the one done in lecture. Another, more generally applicable method, is as follows. First, compute that the eigenvalues of A are 1 and -1. Since the eigenvalues are distinct, A is diagonalizable. Thus, $A = QDQ^{-1}$, where D is the diagonal matrix with diagonal entries $1, -1$ and Q is the matrix whose columns are the two eigenvectors. Clearly $D^{100} =$ Identity and so (as was explained in lecture), $A^{100} = QD^{100}Q^{-1} =$ Identity. Then $A^{101} = A$.

2. Let A_ϵ be the matrix $A_\epsilon = \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where ϵ is a real number.

(a) For what values of ϵ is A_ϵ orthogonal?

(b) Assume $\epsilon \neq 0$. Find the eigenvalues and corresponding eigenspaces of A_ϵ.

(c) Assume $\epsilon \neq 0$. Show that A_ϵ is not diagonalizable.

(d) Let ϵ be any real number. Define the function $F : \mathbb{R} \to \mathbb{R}$ by letting $F(\epsilon)$ equal the dimension of the eigenspace of A_ϵ corresponding to eigenvalue 1. Is F continuous at $\epsilon = 0$? Hint: use (b) to find $\lim_{\epsilon \to 0} F(\epsilon)$, and compare this to $F(0)$.

Solution.

(a) There are a couple of ways one can do the first part.

Method 1. The columns of an orthogonal matrix are orthonormal vectors. Note that the second column of A_ϵ has square length $1 + \epsilon^2$, so this equals one if and only if $\epsilon = 0$, in which case A_ϵ is the identity, an orthogonal matrix. Thus, A_ϵ is orthogonal if and only if $\epsilon = 0$.

Method 2. A_ϵ is orthogonal when $A_\epsilon^T A_\epsilon = I$. Calculate

$$A_\epsilon^T A_\epsilon = \begin{bmatrix} 1 & \epsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+\epsilon^2 & \epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence A_ϵ is orthogonal if and only if ϵ is zero.

(b) $\det(\lambda I - A_\epsilon) = (\lambda - 1)^3$, so A_ϵ has the single eigenvalue $\lambda = 1$. The corresponding eigenspace is

$$\ker(I - A_\epsilon) = \ker \begin{bmatrix} 0 & -\epsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

since $\epsilon \neq 0$.

(c) We see in (b) that for $\epsilon \neq 0$, A_ϵ does not have a basis of eigenvectors. Therefore A_ϵ is not diagonalizable.

(d) From (b) we know that the eigenspace corresponding to eigenvalue 1 is $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $\epsilon \neq 0$; hence $F(\epsilon) = 2$ for $\epsilon \neq 0$. So $\lim_{\epsilon \to 0} F(\epsilon) = 2$. To calculate $F(0)$, note that A_0 is just the identity matrix which has a full basis of eigenvectors (for instance, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis of eigenvectors). So $F(0) = 3$. Since $\lim_{\epsilon \to 0} F(\epsilon) = 2 \neq 3 = F(0)$, F is not continuous at $\epsilon = 0$.

3. Let $g(x, y, z)$ be a smooth function defined on the whole of $\mathbb{R}^3$.

(a) If ∇g has negative x component in the half space $x \geq 0$, must $g(1, 3, 8)$ be bigger than $g(2, 3, 8)$, and must $g(0, 2, 8)$ be bigger than $g(3, 3, 8)$? In each case prove or give a counter-example.

Let $f(x, y) = x^3 + 3y^2 - 2x^2 y + 6$.

(b) Find ∇f at $(2, 3)$.

(c) Find the equation of the tangent plane to the graph of f at the point $(x, y) = (2, 3)$.

(d) In what direction is the function f decreasing the most, and what is the rate of decrease in that direction?

(e) What is the rate of decrease of f in the direction $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$?

Solution.

(a) i. If ∇g has negative x component for $x \geq 0$ then the one-dimensional function $h(x) = g(x, 3, 8)$ has negative derivative for $x \geq 0$ and thus is decreasing. [*Extra:* recall from one variable calculus that one proves this by the mean value theorem (e.g. by assuming $h(1) \leq h(2)$ and getting a contradiction)]. Thus, as g is decreasing, $g(1, 3, 8) > g(2, 3, 8)$.

ii. Let $g = (-x, 10y, 0)$, then $\nabla g = (-1, 10, 0)$. Note, however, that $g(0, 2, 8) = 20 < g(3, 3, 8) = 27$, and so we have a counterexample. There are many other possible counterexamples.

(b) $\nabla f = (3x^2 - 4xy, 6y - 2x^2)$ and thus $\nabla f(2, 3) = (-12, 10)$.

(c) The equation for the tangent plane at (x_0, y_0) is

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

So given that $\nabla f(2, 3) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, we have

$$z = 17 - 12(x - 2) + 10(y - 3)$$

So the tangent plane equation is

$$z = 11 - 12x + 10y$$

(d) f is decreasing the fastest in the direction $-\nabla f(2, 3) = (12, -10)$. The rate of decrease is $|| - \nabla f(2, 3)|| = \sqrt{12^2 + (-10)^2} = \sqrt{244} = 2\sqrt{61}$.

(e) Since the vector is already normalized, this is determined by computing

$$\nabla f(2, 3). \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{-12 - 10}{\sqrt{2}} = -11\sqrt{2}$$

So $11\sqrt{2}$ is the rate of decrease.

4. Consider the functions A and B defined on $\mathbb{R}^3$ as follows:

$$A(x, y, z) = x^2 + xy - \sin(xy), \;\; B(x, y, z) = x^2y^2\cos(z^2).$$

(a) Suppose x and y are functions of two other variables, r and s. Write down the partials of A and B with respect to r and s in matrix notation.

(b) Simplify the your answer in (a) if $x = s + r$, $y = sr$, $z = s - r$.

Let $f(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 - 8$.

(c) Find the critical points of f.

(d) Characterize the critical points in terms of maxima, minima, or saddle points.

Solution.

(a) Let F be the mapping $(r, s) \mapsto (A, B)$. Then by the chain rule,

$$\begin{aligned}
\mathbf{D}F &= \begin{bmatrix} \frac{\partial A}{\partial s} & \frac{\partial A}{\partial r} \\ \frac{\partial B}{\partial s} & \frac{\partial B}{\partial r} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial A}{\partial x} & \frac{\partial A}{\partial y} & \frac{\partial A}{\partial z} \\ \frac{\partial B}{\partial x} & \frac{\partial B}{\partial y} & \frac{\partial B}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial r} \end{bmatrix} \\
&= \begin{bmatrix} 2x + y - y\cos xy & x - x\cos xy & 0 \\ 2xy^2\cos z^2 & 2x^2 y\cos z^2 & -2x^2y^2 z\sin z^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial r} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial r} \end{bmatrix}
\end{aligned}$$

(b) Putting $x = s + r$, $y = sr$, and $z = s - r$, we see that $\mathbf{D}F$ equals AB, where A is

$$\begin{bmatrix} 2(s+r) + sr - sr\cos[(s+r)sr] & (s+r) - (s+r)\cos[(s+r)sr] & 0 \\ 2(s+r)s^2r^2\cos(s-r)^2 & 2(s+r)^2 sr\cos(s-r)^2 & -2(s+r)^2s^2r^2(s-r)\sin(s-r)^2 \end{bmatrix}$$

and B is

$$\begin{bmatrix} 1 & 1 \\ r & s \\ 1 & -1 \end{bmatrix}$$

If one were to multiply this out, one would get, for example,

$$\begin{aligned}
\frac{\partial A}{\partial s} &= 2(s+r) + sr - sr\cos[(s+r)sr] + r[(s+r) - (s+r)\cos[(s+r)sr]] \\
&= 2s + 2r + 2sr + r^2 - r^2\cos[(s+r)sr].
\end{aligned}$$

(c) The critical points are found by setting the partial derivatives of f equal to zero; this gives the system

$$\begin{aligned} 2x - 3y + 5 &= 0 \\ -3x + 12y - 2 &= 0 \end{aligned}$$

Solving this system, we see that the only critical point is $(-18/5, -11/15)$.

(d) The above critical point is a local (in fact, global) minimum, as can be seen from the fact that the Hessian is positive definite. The Hessian is

$$\frac{1}{2}\begin{bmatrix} 2 & -3 \\ -3 & 12 \end{bmatrix}$$

The top left entry is $\partial^2 f/\partial x^2 = 2 > 0$ and the determinant is $15 > 0$, so the Hessian is indeed positive definite.

5. (a) Find the maxima and minima of the function

$$f(x, y) = 2x^2y - x^2 - y^2.$$

on the region $x^2 + y^2 \leq 1$.

(b) Find the extrema of $f(x, y) = xy$ subject to the three conditions $2x + 3y \leq 10, 0 \leq x, 0 \leq y$.

(c) Consider the function $f(x, y) = a\,x^2 + 2b\,xy + c\,y^2$. Suppose that the eigenvalues of the matrix

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

are both positive. Must the origin be a minimum of f?

Solution.

(a) The critical points of f are found by solving the system

$$\begin{aligned} \frac{\partial f}{\partial x} &= 4xy - 2x = 0 \\ \frac{\partial f}{\partial y} &= 2x^2 - 2y = 0, \end{aligned}$$

which gives the critical points $(0,0)$ and $(\pm 1/\sqrt{2}, 1/2)$. Both points are inside the unit circle. The second derivative matrix is

$$\begin{bmatrix} 4y-2 & 4x \\ 4x & -2 \end{bmatrix}$$

which is negative definite at the origin (so the origin is a local maximum, with $f(0,0) = 0$) and it is indefinite at the points $(\pm 1/\sqrt{2}, 1/2)$, which are therefore saddle points.

One must now examine the behavior of f on the boundary using Lagrange multipliers. The Lagrange multiplier conditions ($\nabla f = \lambda \nabla g$ and $g = 0$), give (after canceling factors of 2):

$$\begin{aligned} 2xy - x &= \lambda x \\ x^2 - y &= \lambda y \\ x^2 + y^2 - 1 &= 0 \end{aligned}$$

If $x = 0$, the solution is $(0, \pm 1)$ and if $x \neq 0$, the solution is found by canceling x in the first equation, solving for y and substituting into the remaining equations. This gives $\lambda = \pm 2/\sqrt{3} - 1$, $x = \pm\sqrt{2/3}$ and $y = \pm 1/\sqrt{3}$. Evaluating f at these points gives the answer: $(0,0)$ is a maximum and the minimum value of f is $(-9 - 4\sqrt{3})/9$, which occurs at the points $(\pm\sqrt{2/3}, -1/\sqrt{3})$ on the boundary.

(b) The region on which f is defined is the triangle in the first quadrant of the xy-plane bounded by the axes and the line $2x+3y = 10$. Since $f_x = y$, $f_y = x$, we see there is no critical point strictly inside this region. Thus, the extrema must be on the boundary.

The boundary of this triangular region consists of 3 straight line segments, and we need to find the absolute maximum and minimum of f on this boundary. Clearly $f = 0$ on both $x = 0$ and $y = 0$; since $f \geq 0$ in the region, the minimum value of 0 occurs on the two axes. To find the maximum, we only need to solve the following Lagrange multiplier system with $x > 0$ and $y > 0$:

$$\begin{aligned} y &= 2\lambda \\ x &= 3\lambda \\ 2x + 3y &= 10 \end{aligned}$$

Solving these gives $\lambda = \frac{5}{6}, x = \frac{5}{2}, y = \frac{5}{3}$. Also, $f(\frac{5}{2}, \frac{5}{3}) = \frac{25}{6}$. In summary, the absolute maximum value is $\frac{25}{6}$ which occurs at the point $(5/2, 5/3)$ and the absolute minimum value of 0 occurs at points on the two axes.

(c) Yes. Call the eigenvalues λ and μ. The condition that $\lambda > 0$ and $\mu > 0$ is equivalent to the positive definiteness of the Hessian, which is half of the second derivative matrix,

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

One should relate this to the second derivative conditions given in the book, namely that $a > 0$ and $ac - b^2 > 0$. One way to do this is as follows. Taking the trace and determinant of A (recall that the trace is the sum of the eigenvalues and the determinant is their product), we get

$$\begin{aligned} \lambda + \mu &= a + c \\ \lambda\mu &= ac - b^2 \end{aligned}$$

From this, it is clear that the conditions $a > 0$ and $ac - b^2 > 0$ are equivalent to the conditions $\lambda > 0$ and $\mu > 0$.